A Banach space dichotomy theorem for quotients of subspaces

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Abstract

A Banach space X with a Schauder basis is defined to have the *restricted quotient hereditarily indecomposable property* if X/Y is hereditarily indecomposable for any infinite codimensional subspace Y with a successive finite-dimensional decomposition on the basis of X. The following dichotomy theorem is proved: any infinite dimensional Banach space contains a quotient of subspace which either has an unconditional basis, or has the restricted quotient hereditarily indecomposable property. [1]

1 Introduction

In 2002, W.T. Gowers published his famous Ramsey theorem for block-subspaces in a Banach space [8]. If X is a Banach space with a Schauder basis, *block-vectors* in X denote non zero vectors with finite support on the basis, and *block-sequences* are infinite sequences of block-vectors with successive supports; *block-subspaces* are subspaces generated by block-sequences.

If Y is a block-subspace of X, *Gowers' game in* Y is the infinite game where Player 1 plays block-subspaces Y_n of Y, and Player 2 plays normalized block vectors y_n in Y_n.

If $\Delta = (\delta_n)_{n\in\mathbb{N}}$ is a sequence of reals, $\Delta > 0$ means that $\delta_n > 0$ for all $n \in \mathbb{N}$. For A a set of normalized block-sequences, and any $\Delta = (\delta_n)_{n\in\mathbb{N}} > 0$, let A_Δ be the set of normalized block-sequences $(y_n)_{n\in\mathbb{N}}$ such that there exists $(x_n)_{n\in\mathbb{N}}$ in A with $\|x_n - y_n\| \leq \delta_n$ for all $n \in \mathbb{N}$.

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Theorem 1 *(Gowers' Ramsey Theorem) Let A be a set of normalized block-sequences which is analytic as a subset of X^ω with the product of the norm topology on X. Assume that every block-subspace of X contains a block-sequence in A. Let $\Delta > 0$. Then there exists a block-subspace Y of X such that Player 2 has a winning strategy in Gowers' game in Y for producing a sequence $(y_n)_{n\in\mathbb{N}}$ in A_Δ.*

The most important consequence of the Ramsey Theorem of Gowers is the so-called dichotomy theorem for Banach spaces. A Banach space X is said to be *decomposable* if it is a direct (topological) sum of two infinite-dimensional closed subspaces. An infinite dimensional space is *hereditarily indecomposable (or HI)* when it has no decomposable subspace. A Schauder basis $(e_n)_{n\in\mathbb{N}}$ of X is unconditional if there exists $C \geq 1$ such that for all $\sum_{i\in\mathbb{N}} \lambda_i e_i$ in X, all $(\epsilon_i)_{i\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$, $\left\|\sum_{i\in\mathbb{N}} \epsilon_i \lambda_i e_i\right\| \leq C \left\|\sum_{i\in\mathbb{N}} \lambda_i e_i\right\|$.

Theorem 2 *(Gowers' Dichotomy Theorem) Every infinite dimensional Banach space contains a subspace Y which satisfies one of the two following properties, which are both possible, and mutually exclusive:*
i) Y has an unconditional basis,
ii) Y is hereditarily indecomposable.

These properties are even exclusive in the sense that if a space satisfies i) (resp. ii)), then no further subspace satisfies ii) (resp. i)). Indeed if a Banach space X is hereditarily indecomposable, then so is any subspace of X; and if X has an unconditional basis, then every block-subspace of X has an unconditional basis, and so any subspace of X has a further subspace with an unconditional basis.

1.1 HI spaces and their quotient spaces

From now on, spaces and subspaces are supposed infinite dimensional and closed unless specified otherwise. For two subspaces Y and Z of a space X, a convenient notion of angle was used by B. Maurey to give a simple proof of Gowers' dichotomy theorem [12]: let

$$a(Y,Z) = \inf_{y\in Y, z\in Z, y\neq z} \frac{\|y-z\|}{\|y+z\|}.$$

It is in particular clear that $a(Y,Z) \neq 0$ if and only if $Y+Z$ forms a topological direct sum in X, and therefore a space X is hereditarily indecomposable if and

only if $a(Y,Z) = 0$ for any subspaces Y, Z of X. On the other hand, a basic sequence $(e_i)_{i\in\mathbb{N}}$ is C-unconditional if $a([e_i, i \in I], [e_i, i \in J]) \geq 1/C$ for every partition $\{I, J\}$ of $\mathbb{N}$, where $[e_i, i \in I]$ denotes the closed linear space generated by $(e_i)_{i\in I}$.

We also note that it was proved in [9] that hereditarily indecomposable spaces are never isomorphic to proper subspaces.

While classical spaces, such as c_0 and $\ell_p, 1 \leq p < +\infty$, or $L_p, 1 < p < +\infty$, have unconditional bases, the first known example of a HI space was given by Gowers and Maurey in 1993, [9]. Gowers-Maurey's space X_{GM} is actually *quotient hereditarily indecomposable (or QHI)*, that is, no quotient of a subspace of X_{GM} is decomposable, or equivalently, every infinite dimensional quotient space of X_{GM} is HI [6]; as X_{GM} is reflexive, it follows that X^*_{GM} is also quotient hereditarily indecomposable, and in particular also hereditarily indecomposable. In [6], an example X was also provided which is HI and not QHI. This example is defined as the "push-out" $(X_1 \oplus X_2)/\{(y, -y) : y \in Y\}$ of two specific Gowers-Maurey's type spaces X_1 and X_2 with respect to a "common" subspace Y. It is therefore still very close to being QHI, in the sense that it is saturated with QHI subspaces, and the natural quotient space of X which is decomposable is a direct sum of two HI spaces. This led the author to conjecture that any quotient of a HI space should contain a HI or even QHI subspace, or that the dual of any reflexive HI space should contain a HI subspace.

This however turned out to be completely false. Examples of HI spaces were built with quotients which are very far from being HI. Using methods based on the definition of some notion of HI interpolation of Banach spaces, S. Argyros and V. Felouzis constructed a HI space with some quotient space isomorphic to c_0 (resp. $l_p, 1 < p < +\infty$) [2]. S. Argyros and A. Tolias used deep constructions, based on what is now known as the "extension method" [1], to prove that any separable Banach space which does not contain a copy of l_1 is isomorphic to the quotient space of some separable HI space [4]; and to construct a reflexive Banach space X which is HI but whose dual is saturated with unconditional basic sequences [5], therefore any quotient space of X has a further quotient with an unconditional basis. These results shatter all hopes of general results preserving the HI property when passing to quotient spaces, or to the dual. We refer to [3], [4], and [11] for more details about these examples and hereditarily indecomposable spaces in general, as well as about other examples, and also to the recent work [1] which contains an comprehensive introduction to the previous examples.

S. Argyros asked whether there existed a reflexive HI Banach space X, such that no subspace of X has a HI dual. This would show that the H.I. structure is in general not inherited by duals, not even in a very weak sense. None of the HI examples constructed so far seem to answer that question (for more about this, we refer to the remarks and questions section at the end of this paper).

Our main result is somewhat related to the question of Argyros. Its starting point is the observation that the situation becomes more pleasant again when one looks at quotient of subspaces (or QS-spaces) of a given Banach space. First note that the features of the QHI property with respect to quotient of subspaces are quite similar to the ones of the HI property with respect to subspaces. Indeed this property obviously passes to further QS-spaces. We also have the following result.

Proposition 3 *If X is hereditarily indecomposable, then X is isomorphic to no proper quotient of subspace of itself.*

Proof : Assume X is HI and α is an isomorphism from X onto Y/Z for some $Z \subset Y \subset X$. We may assume that $\dim Z = +\infty$. Then by properties of HI spaces [12], the quotient map $\pi : Y \rightarrow Y/Z$ is strictly singular. The map $T = \alpha\pi$ is an onto map whose Fredholm index $i(T)$ (defined as $\dim(KerT) - \dim(X/TY)$ when this expression has a meaning) is $+\infty$. By continuity of the index ([10] Proposition 2.c.9), we deduce that $i(T - \epsilon i_{YX}) = +\infty$ for some small enough $\epsilon > 0$. On the other hand, T is strictly singular, therefore, by [10] Proposition 2.c.10, $i(T - \epsilon i_{YX}) = i(-\epsilon i_{YX}) \leq 0$. $\square$

The unconditional property also satisfies some type of heredity for quotient of subspaces. T. Odell proved that if X has a shrinking finite-dimensional unconditional decomposition, then every normalized weakly null sequence in a quotient of X has an unconditional subsequence [13], and therefore every QS-space of X contains an unconditional basic sequence.

It is therefore tempting to look for some general dichotomy result for quotient of subspaces involving the QHI property on one side and some unconditionality property on the other.

1.2 Angles between quotient of subspaces

To motivate our following definitions and results, we take a closer look at Gowers-Maurey's sequence space X_{GM}. To prove that X_{GM} is HI, Gowers and Maurey build, for arbitrary large $k \in \mathbb{N}$, successive biorthonormal sequences $(y_i)_{i\leq k}$ and $(y_i^*)_{i\leq k}$ of "special" pairs of vectors and functionals, such that

$$\left\| \sum_{i\leq k} y_i^* \right\| \simeq \sqrt{\log(k)},$$

while

$$\left\| \sum_{i\leq k} (-1)^i y_i \right\| \simeq k/\log(k).$$

Up to a perturbation, the terms $(y_i)_{i\leq k}$ may be taken in arbitrary subspaces of X_{GM}. Therefore, given $Y, Z \subset X_{GM}$, by taking the even terms close enough to Y and the odd terms close enough to Z, we may find vectors y almost in Y and z almost in Z, and functionals y^* and z^*, with disjoint supports, such that $\|y - z\| \simeq k/\log(k)$ while

$$\|y + z\| \geq \frac{(y^* + z^*)(y + z)}{\|y^* + z^*\|} \simeq k/\sqrt{\log(k)}.$$

It follows that $Y + Z$ is never a direct sum.

The proof in [6] that X_{GM} is QHI is based on the fact that one can actually choose y^* and z^* close enough to $W^\perp$ for any W which is an infinite codimensional subspace of Y and of Z. It follows easily that X_{GM} is quotient hereditarily indecomposable. By the proof it is clear than one can even pick y^* close enough to $V^\perp$ and z^* close enough to $W^\perp$ for any infinite codimensional subspaces V of Y and W of Z. The point here is that each term of the sequences of "special" vectors (resp. functionals) must be taken in some set A_n (resp. A_n^*) which is asymptotic, i.e. intersects any subspace of X_{GM} (resp. X_{GM}^*), associated to some n depending on the previous terms, but the subspace in which to pick it may be chosen arbitrarily.

For X a Banach space, and a subspace Y_* of X^*, denote by $\|.\|_{Y_*}$ the semi-norm defined on X by $\|x\|_{Y_*} = \sup_{y^* \in Y_*, \|y^*\| \leq 1} y^*(x)$, and by $Y_*^\perp$ the orthogonal of Y_* in X. When $Y_* = Y^\perp$ for some $Y \subset X$, $\|.\|_{Y_*}$ is the quotient norm on X/Y.

A *QS-pair* is some $(Y_*, Y) \subset X^* \times X$ such that $Y_*^\perp \subset Y$. It may be associated to the QS-space $Y/Y_*^\perp$. The natural notion of inclusion between QS-pairs

$$(Z_*, Z) \subset (Y_*, Y) \Leftrightarrow (Z_* \subset Z) \wedge (Y_* \subset Y)$$

corresponds to taking quotient of subspaces of the associated QS-spaces. Indeed if $(Z_*, Z) \subset (Y_*, Y)$, then $Z/Z_*^\perp \simeq (Z/Y_*^\perp)/(Z_*^\perp/Y_*^\perp)$. An *infinite dimensional QS-pair* is a QS-pair whose associated QS-space is infinite dimensional. We define the angle $A((Y_*, Y), (Z_*, Z))$ between two QS-pairs by

$$A((Y_*, Y), (Z_*, Z)) = \inf_{y \neq z, y^* \neq z^*, y^*(z) = z^*(y) = 0} \frac{\|y - z\| \, \|y^* - z^*\|}{|y^*(y) - z^*(z)|},$$

where the infimum is taken over $y \in Y, z \in Z, y^* \in Y_*, z^* \in Z_*$.

Note that if we let $W_* = Y_* = Z_*$, then we obtain

$$A((W_*, Y), (W_*, Z)) \geq \inf_{y \neq z, \|y^* - z^*\| = 1} \frac{\|y - z\|}{|(y^* - z^*)(y + z)|} \geq \inf_{y \neq z} \frac{\|y - z\|_{W_*}}{\|y + z\|_{W_*}},$$

and therefore when $W_* = W^\perp$ for some $W \subset X$, $A((W_*, Y), (W_*, Z)) \geq a(Y/W, Z/W)$. In particular Y/W and Z/W do not form a direct sum in X/W when $A((W^\perp, Y), (W^\perp, Z)) = 0$. If this is true for all W, Y, Z with W an infinite codimensional subspace of Y and of Z then we deduce that X is QHI.

By our previous description of special sequences in Gowers-Maurey's space, X_{GM} is an exemple of a reflexive space for which $A((Y_*, Y), (Z_*, Z)) = 0$ for all infinite dimensional QS-pairs (Y_*, Y) and (Z_*, Z) of X. Indeed if y, z, y^*, $-z^*$ are the odd and even parts respectively of adequate length k special sequences, we have

$$\|y - z\| \, \|y^* - z^*\| \simeq k/\sqrt{\log(k)},$$

while

$$|y^*(y) - z^*(z)| \simeq k.$$

By construction, we may pick the terms of the special sequences close enough to Y, Z, Y_*, Z_* respectively. It is not difficult to check that we may then perturb the almost biorthonormal system of special sequences in such a way as to assume that $y \in Y$, $z \in Z$, $y^* \in Y_*$, $z^* \in Z_*$, and $y^*(z) = z^*(y) = 0$, and preserving the estimates on $\|y - z\| \, \|y^* - z^*\|$ and $|y^*(y) - z^*(z)|$.

When X is reflexive, the roles of X and X^* are interchangeable in the expression of A. Note that under reflexivity, the QHI property ([6] Corollary 4) and the property of having an unconditional basis are self-dual properties.

1.3 FDD-block subspaces and FDD-block quotient of subspaces

We shall prove that a dichotomy theorem holds for quotient of subspaces which have a finite-dimensional decomposition (or FDD) relative to a given Schauder basis (or even a FDD) of a given Banach space; they seem to be the natural equivalent of block-subspaces considered in Gowers' dichotomy.

An *interval of integers* is the intersection of $\mathbb{N}$ with a bounded interval of $\mathbb{R}$. Two non empty intervals E_1 and E_2 are said to be *successive*, $E_1 < E_2$, when $\max(E_1) < \min(E_2)$. A *successive partition* will be a sequence $(E_n)_{n\in\mathbb{N}}$ of successive intervals forming a partition of $\mathbb{N}$.

Let X be a Banach space with a finite-dimensional decomposition denoted $(B_n)_{n\in\mathbb{N}}$. When $x = \sum_{n\in\mathbb{N}} b_n \in X$, with $b_n \in B_n$ for all $n \in \mathbb{N}$, the *support of* x is the set $\{i \in \mathbb{N} : b_i \neq 0\}$. The *range* of a vector is the smallest interval containing its support. The support of a subspace Y of X is the smallest set containing the supports of all vectors of Y. The range of Y is the smallest interval containing the support of Y. Two finitely supported subspaces F and G of X with non-empty supports are *successive* when $ran(F) < ran(G)$.

An *FDD-block subspace* of X is an infinite sum $\sum_{n\in\mathbb{N}} F_n$ of finitely supported (possibly zero-dimensional) subspaces F_n of X, such that $ran(F_n) \subset E_n, \forall n \in \mathbb{N}$, where $(E_n)_{n\in\mathbb{N}}$ is a successive partition. Therefore an FDD-block subspace is finite-dimensional or equipped with the FDD $(F_n)_{n\in I}$, where $I = \{n : F_n \neq \{0\}\}$.

An *FDD-block quotient* of X is the quotient of X by some FDD-block subspace $Y = \sum_{n\in\mathbb{N}} G_n$. An FDD-block quotient is finite-dimensional or equipped with the FDD $(C_n)_{n\in I}$ corresponding to the successive partition $(E_n)_{n\in\mathbb{N}}$ associated to Y, that is, where $C_n = ([B_i, i \in E_n] + Y)/Y$ for all n, and $I = \{n : C_n \neq \{0\}\}$. Note that the space X is an FDD-block quotient of itself.

An *FDD-block quotient of subspace* of X is a quotient of subspace of X of the form $\sum_{n\in\mathbb{N}} F_n / \sum_{n\in\mathbb{N}} G_n$, where $G_n \subset F_n \subset [B_i, i \in E_n]$ for all n, where $(E_n)_{n\in\mathbb{N}}$ is a successive partition. The space $\sum_{n\in\mathbb{N}} F_n / \sum_{n\in\mathbb{N}} G_n$ is naturally seen

as an FDD-block subspace of $X/\sum_{n\in\mathbb{N}} G_n$, when $X/\sum_{n\in\mathbb{N}} G_n$ is equipped with the FDD corresponding to $(E_n)_{n\in\mathbb{N}}$.

It is therefore clear that any FDD-block subspace (resp. quotient of subspace) of an FDD-block subspace (resp. quotient of subspace) of X is again an FDD-block subspace (resp. quotient of subspace) of X. Note also that by classical results, any subspace of X contains, for any $\epsilon > 0$, an $1+\epsilon$-isomorphic copy of a block subspace, and therefore of an FDD-block subspace (however the similar result concerning QS-spaces doesn't seem to be clear). Considering FDD-block quotient of subspaces to study the structure of the class of QS-spaces is a natural counterpart of considering block-subspaces to study the structure of the class of subspaces.

Proposition 4 *Let X be a Banach space with a finite-dimensional decomposition. The following propositions are equivalent:*

i) no FDD-block quotient of subspace of X is decomposable,

ii) for any infinite codimensional FDD-block subspace Y of X, the quotient X/Y is hereditarily indecomposable,

iii) whenever $Y = \sum_{n\in\mathbb{N}} F_n/\sum_{n\in\mathbb{N}} G_n$ and $Y' = \sum_{n\in\mathbb{N}} F'_n/\sum_{n\in\mathbb{N}} G_n$ are infinite dimensional FDD-block quotient of subspaces of X with a same successive partition, the sum $Y+Y'$ is not direct in $X/(\sum_{n\in\mathbb{N}} G_n)$.

When X satisfies i) ii) iii) we shall say that X is quotient hereditarily indecomposable restricted to FDD-block subspaces, or in short, has the restricted QHI property.

Proof : ii) implies i) is immediate. If iii) is false then the FDD-block quotient of subspace $Y+Y' = \sum_{n\in\mathbb{N}}(F_n+F'_n)/\sum_{n\in\mathbb{N}} G_n$ is decomposable, contradicting i). Finally, assume ii) is false, i.e. $Z/W \oplus Z'/W$ forms a direct sum of infinite dimensional subspaces in X/W, for some infinite codimensional FDD-block subspace $W = \sum_{n\in\mathbb{N}} G_n$ and some subspaces Z and Z', and let $(E_n)_{n\in\mathbb{N}}$ be a successive partition associated to W. We may up to a perturbation find sequences $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$, and a partition $(N_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ into successive intervals, such that for all $n \in \mathbb{N}$, $ran(z_n, z'_n) \subset \cup_{i\in N_n} E_i$, and such that $d(z_n, Z)$ and $d(z'_n, Z')$ converge to 0 sufficiently fast so that $([z_n]_{n\in\mathbb{N}} + W)/W \oplus ([z'_n]_{n\in\mathbb{N}} + W)/W$ is still direct in X/W. Let for all $n \in \mathbb{N}$, $H_n = \sum_{i\in N_n} G_i$; we have therefore obtained that $(\sum_{n\in\mathbb{N}}(H_n + [z_n]))/\sum_{n\in\mathbb{N}} H_n$ and $(\sum_{n\in\mathbb{N}}(H_n + [z'_n]))/\sum_{n\in\mathbb{N}} H_n$ form a direct sum, with successive partition $(\cup_{i\in N_n} E_i)_{n\in\mathbb{N}}$, contradicting iii). $\square$

FDD-block quotient of subspaces still capture enough information about the structure of the space: a space which has the restricted QHI property is in particular hereditarily indecomposable by ii), and by i), any of its infinite-dimensional FDD-block quotient of subspaces has again the restricted QHI property. The next proposition also shows that the restricted QHI property has similar self-dual properties as the QHI property.

Proposition 5 *Let X be a Banach space with a shrinking finite-dimensional decomposition, such that X^* has the restricted QHI property. Then X has the restricted QHI property.*

Proof : Let $Y = \sum_{n\in\mathbb{N}} F_n / \sum_{n\in\mathbb{N}} G_n$ be an infinite dimensional FDD-block quotient of subspace of X, with successive partition $(E_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let X_n^* be the space of vectors in X^* with range included in E_n. Then

$$Y^* = (\sum_{n\in\mathbb{N}} G_n)^\perp / (\sum_{n\in\mathbb{N}} F_n)^\perp = (\sum_{n\in\mathbb{N}} (G_n^\perp \cap X_n^*)) / (\sum_{n\in\mathbb{N}} (F_n^\perp \cap X_n^*)),$$

since $(E_n)_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$. So Y^* is an FDD-block quotient of subspace of X^*. Therefore according to the first characterization in Proposition 4, if X does not have the restricted QHI property, then X^* does not have the restricted QHI property. $\square$

In consequence, we note that if X is a reflexive Banach space with the restricted QHI property, then X has HI dual and X is saturated with subspaces with HI dual. Indeed every FDD-block subspace of X has HI dual.

We are now in position to state the result of this paper.

Theorem 6 *Every Banach space has a quotient of subspace Y with one of the two following properties, which are mutually exclusive and both possible:*
i) Y has an unconditional basis,
ii) Y has the restricted QHI property.

We give a few comments on the reasons we needed to impose a restriction on the QHI property. Our proof is based on some method of ”combinatorial forcing”, see Todorcevic’s course [3] about this. This will enable us to prove, up to some approximation, a general dichotomy result for closed properties of FDD-block

quotient of subspaces, seen as sequences of finite dimensional successive QS-blocks (this will be defined precisely in the next section), with the product of the discrete topology on the set of QS-blocks. This applies more or less directly to obtain Theorem 6.

As we see them, these methods rely on defining infinite sequences of elements which may be correctly approximated by finite sequences; a notion of successivity is needed, i.e. finite sequences are extended in infinite sequences in a way that does not ”affect” the properties implied by the finite part.

Our proof was inspired by a simplification by B. Maurey of this method in the case of block-subspaces of a space with a Schauder basis, where a less restrictive setting may be used, based on replacing X by a countable dense subset [12].

It didn't seem possible to repeat exactly Maurey's proof to study QS spaces. Therefore we needed to restrict our study to particular QS spaces which may be canonically associated to infinite sequences of ”finite dimensional blocks” which are successive in some sense. For technical reasons, the countable dense subset must be replaced by a net whose intersection with the set of predecessors of a given block is always finite. Up to perturbations, the restriction to a net is not essential, but the need for some notion of successivity seems to be, and this justifies that we could not obtain ”quotient hereditarily indecomposable” in the second part of the conclusion of Theorem 6. Actually some examples indicate that FDD-block quotient of subspaces may behave differently from general quotient of subspaces. We refer to the final section about this fact.

2 Proof of the theorem

To prove Theorem 6, we may consider a Banach space X with a Schauder basis (e_n). We denote by (e_n^*) its dual basis, and by X_* the closed linear span of $(e_n^*)_{n\in\mathbb{N}}$. We may also assume, up to renorming, that the basis is bimonotone. We shall consider supports and ranges of vectors, or subspaces, of X and of X_*, with respect to these canonical bases.

We choose to represent blocks forming quotient of subspaces of X as pairs formed by a finite-dimensional subspace F of X and of a finite dimensional subspace of F_* of X_*, with $F_*^\perp \cap [e_n, n \in ran(F, F_*)] \subset F$. Pairs (F, G) of finite-dimensional subspaces of X with $G \subset F$ would also have been a possible representation. Our choice will save us from some technicalities (successive pairs in our setting are pairs whose supports are necessarily immediately successive). It will also preserve, in our proofs, the symmetry between the roles played by X and

X^* in the reflexive case. This symmetry is apparent in our main result, and we felt it worth to be emphasized in our demonstration.

2.1 Blockings of QS-pairs

If $Y \subset X$, and $Y_* \subset X_*$, the range of (Y_*, Y) is the smallest interval containing the ranges of Y and of Y_*. The set of finitely supported subspaces of X is denoted $F(X)$, of finitely supported subspaces of X_* is denoted $F(X_*)$. A *QS-block (or block)* is a pair $(F_*, F) \in F(X_*) \times F(X)$, therefore $E := ran(F_*, F)$ is finite, such that $F_*^\perp \cap [e_n, n \in E] \subset F$. The set of blocks is denoted $\mathcal{F}(X)$. The *dimension* of (F_*, F) is the dimension of $F/(F_*^\perp \cap [e_n, n \in E])$. Two blocks (F_*, F) and (G_*, G) are said to be *successive* if $\min(ran(G_*, G)) = \max(ran(F_*, F)) + 1$, and we write $(F_*, F) < (G_*, G)$ (note the technical difference with the usual notion of successivity).

We note that when $\mathcal{Y} = (Y_{n*}, Y_n)_{n \in \mathbb{N}}$ is a sequence of successive blocks whose ranges partition $\mathbb{N}$ (equivalently, such that $\min(ran(Y_{1*}, Y_1) = 1)$, the spaces $Y = \sum_{n \in \mathbb{N}} Y_n$ and $Y_* = \sum_{n \in \mathbb{N}} Y_{n*}$ satisfy $Y_*^\perp \subset Y$. We shall then say that (Y_*, Y) is the *QS-pair associated to* $\mathcal{Y}$, and that $\mathcal{Y}$ is *infinite dimensional* to mean that the QS-space $Y/Y_*^\perp$ is infinite dimensional. Note that the space $Y/Y_*^\perp$ is a block-FDD quotient of subspace of X.

If (F_*, F) is a block and $(Y_{n*}, Y_n)_{n \in I}$ is a finite or infinite sequence of successive blocks, and if there exists an interval $E \subset I$ such that $F \subset \sum_{n \in E} Y_n$ and $F_* \subset \sum_{n \in E} Y_{n*}$, then we shall say that (F_*, F) *is a block of* $(Y_{n*}, Y_n)_{n \in I}$.

We now define a relations of "blocking" between sequences of successive blocks.

Definition 7 *Let $(Y_{n*}, Y_n)_n$ and $(Z_{i*}, Z_i)_i$ be finite or infinite sequences of successive blocks. If for any i, (Z_{i*}, Z_i) is a block of $(Y_{n*}, Y_n)_n$, then we shall say that $(Z_{i*}, Z_i)_i$ is a blocking of $(Y_{n*}, Y_n)_n$.*

If $\mathcal{Z} = (Z_{i*}, Z_i)_{i \in \mathbb{N}}$ and $\mathcal{Y} = (Y_{n*}, Y_n)_{n \in \mathbb{N}}$ are infinite sequences of successive blocks whose ranges partition $\mathbb{N}$, then we shall write $\mathcal{Z} \leq \mathcal{Y}$ to mean that $\mathcal{Z}$ is a blocking of $\mathcal{Y}$. This means that there exists a partition $\{N_i, i \in \mathbb{N}\}$ of $\mathbb{N}$ in successive intervals such that, for all $i \in \mathbb{N}$, $ran(Z_{i*}, Z_i) = \cup_{n \in N_i} ran(Y_{n*}, Y_n)$ and (Z_{i*}, Z_i) is a block of $(Y_{n*}, Y_n)_{n \in N_i}$.

We note that $\leq$ is an order relation. Clearly, when $\mathcal{Z} \leq \mathcal{Y}$, the associated QS-pairs (Z_*, Z) and (Y_*, Y) satisfy $(Z_*, Z) \subset (Y_*, Y)$.

For any two sequences $\mathcal{Y}$ and $\mathcal{Z}$, we define

$$A(\mathcal{Y}, \mathcal{Z}) = A((Y_*, Y), (Z_*, Z)),$$

where (Y_*, Y) and (Z_*, Z) are the associated QS-pairs.

Lemma 8 *Let $\mathcal{Y} = (Y_{n*}, Y_n)_{n\in\mathbb{N}}$ be an infinite dimensional, successive sequence of blocks whose ranges partition $\mathbb{N}$. Let (Y_*, Y) be the associated QS-pair. Assume that $A(\mathcal{U}, \mathcal{V}) = 0$ whenever $\mathcal{U}, \mathcal{V} \leq \mathcal{Y}$ are infinite dimensional, successive sequences of blocks whose ranges are equal and partition $\mathbb{N}$. Then $Y/Y_*^\perp$ has the restricted QHI property.*

Proof : The proof is based on the natural identification between sequences of blocks of $Y/Y_*^\perp$ with its natural finite-dimensional decomposition, and sequences of blocks of X which are blockings of (Y_*, Y). Indeed consider two infinite dimensional block-FDD quotient of subspaces of $Y/Y_*^\perp$ which are of the form $Z = \sum_{n\in\mathbb{N}} F_n / \sum_{n\in\mathbb{N}} G_n$ and $Z' = \sum_{n\in\mathbb{N}} F'_n / \sum_{n\in\mathbb{N}} G_n$, with successive partition $(E_n)_{n\in\mathbb{N}}$. By definition for all $n \in \mathbb{N}$, $F_n \subset (\sum_{k\in E_n} Y_k + Y_*^\perp)/Y_*^\perp$, and let $I_n = ran(\sum_{k\in E_n} Y_k)$. Therefore we may find A_n, B_n such that

$$(\sum_{k\in E_n} Y_{k*})^\perp \cap [e_i, i \in I_n] \subset B_n \subset A_n \subset \sum_{k\in E_n} Y_n,$$

and such that $G_n = (B_n + Y_*^\perp)/Y_*^\perp$ and $F_n = (A_n + Y_*^\perp)/Y_*^\perp$. We define some subspaces A'_n associated to the spaces F'_n in a similar way.

We therefore have the identification

$$Z = \overline{\sum_{n\in\mathbb{N}} A_n + Y_*^\perp} / \overline{\sum_{n\in\mathbb{N}} A_n + Y_*^\perp} = \sum_{n\in\mathbb{N}} A_n / \sum_{n\in\mathbb{N}} B_n,$$

which is by construction a block FDD quotient of subspace of X corresponding to a blocking of $\mathcal{Y}$. Indeed, let $B_{n*} = B_n^\perp \cap [e_i, i \in I_n]$, and let $\mathcal{Z} = (B_{n*}, A_n)_{n\in\mathbb{N}}$, then the associated QS-space is $\sum_{n\in\mathbb{N}} A_n / (\sum_{n\in\mathbb{N}} A_{n*})^\perp = \sum_{n\in\mathbb{N}} A_n / \sum_{n\in\mathbb{N}} B_n$.

We have the similar identification for Z' and let $\mathcal{Z}' = (B_{n*}, A'_n)_{n\in\mathbb{N}}$. Since $\mathcal{Z} \leq \mathcal{Y}$ and $\mathcal{Z}' \leq \mathcal{Y}$, it follows that $A(\mathcal{Z}, \mathcal{Z}') = 0$. This means that the spaces $\sum_{n\in\mathbb{N}} A_n / \sum_{n\in\mathbb{N}} B_n$ and $\sum_{n\in\mathbb{N}} A'_n / \sum_{n\in\mathbb{N}} B_n$ do not form a direct sum in $Y / \sum_{n\in\mathbb{N}} B_n$, and therefore Z and Z' do not form a direct sum in the space $(Y/Y_*^\perp)/(\sum_{n\in\mathbb{N}} G_n)$. Therefore iii) is satisfied in Proposition 4. □

Before stating more definitions, we need to realize a reduction to a net $\mathcal{R}$ of blocks with some finiteness property which will be crucial for our combinatorial method.

For F, G in $F(X)$, we let $d_H(F,G)$ be the Hausdorff distance between the unit spheres S_F of F and S_G of G, $d_H(F,G) = \max_{x \in S_F} d(x, S_G) \vee \max_{y \in S_G} d(y, S_F)$. Modifying a definition from [7], we define a distance d on $F(X)$ by

$$d(F,G) = \min(1, 2k\sqrt{k} d_H(F,G))$$

if $\dim F = \dim G = k$ and $ran(F) = ran(G)$, and $d(F,G) = 1$ otherwise.

We finally define a distance δ on $\mathcal{F}(X)$ by

$$\delta((F_*, F), (G_*, G)) = \max(d(F,G), d(F_*^\perp \cap X_0, G_*^\perp \cap X_0)),$$

when $ran(F_*, F) = ran(G_*, G)$ and $X_0 = [e_i, i \in ran(F_*, F)]$, and we let $\delta((F_*, F), (G_*, G)) = 1$ otherwise.

The critical result concerning this distance is contained in the next lemma.

Lemma 9 *Let $0 < \epsilon < 1$ and let $(\delta_n)_n$ be a positive sequence such that $\sum_{n \in \mathbb{N}} \delta_n \leq \epsilon$. Let $(F_{n*}, F_n)_{n \in \mathbb{N}}$ and $(G_{n*}, G_n)_{n \in \mathbb{N}}$ be successive sequences of blocks such that for all $n \in \mathbb{N}$, $\delta((F_{n*}, F_n), (G_{n*}, G_n)) \leq \delta_n$, and, let, for $n \in \mathbb{N}$, X_n be the space $[e_i, i \in ran(F_{n*}, F_n)]$. Then there exists a map $T : \sum_{n \in \mathbb{N}} F_n \to \sum_{n \in \mathbb{N}} G_n$ such that $T(F_n) = G_n$ and $T(F_{n*}^\perp \cap X_n) = G_{n*}^\perp \cap X_n$ for all $n \in \mathbb{N}$, and such that for any $x \in \sum_{n \in \mathbb{N}} F_n$, $\|Tx - x\| \leq \epsilon \|x\|$.*

Proof : Let $k = \dim F_1 = \dim G_1$ and let $l = \dim F_{1*}^\perp \cap X_1 = \dim G_{1*}^\perp \cap X_1$. By classical results, the Banach-Mazur distance of F_1 to l_2^k is at most $\sqrt{k}$, so we may pick a normalized basis $f_1, \ldots, f_k$ of F_1 such that $f_1, \ldots, f_l$ is a basis of $F_{1*}^\perp \cap X_1$ and which has basis constant at most $\sqrt{k}$. By the expression of δ, we have that $d_H(F_{1*}^\perp \cap X_1, G_{1*}^\perp \cap X_1) \leq \delta_1 / 2k\sqrt{k}$, therefore for $1 \leq i \leq l$, there exists some $g_i \in G_{1*}^\perp \cap X_1$ with $\|g_i - f_i\| \leq \delta_1 / 2k\sqrt{k}$. Likewise we find for $l < i \leq k$ some $g_i \in G_1$ with the same condition on $\|g_i - f_i\|$.

By [10] Prop. 1.a.9, $(g_i)_{1 \leq i \leq k}$ is a basis of G_1, and furthermore, if $T_1 : F_1 \to G_1$ is defined by $T_1(f_i) = g_i$ for all $1 \leq i \leq k$, we have, for any $x \in F_1$, $x = \sum_{i=1}^k a_i f_i$,

$$\|T_1 x - x\| \leq \sum_{i=1}^k |a_i| \|f_i - g_i\| \leq 2\sqrt{k} \|x\| k(\delta_1 / 2k\sqrt{k}) \leq \delta_1 \|x\| .$$

Repeating this construction on each F_n, let T_n be the associated map from F_n onto G_n with $T_n(F_{n*}^\perp \cap X_n) = G_{n*}^\perp \cap X_n$, and let T be defined on $\sum_{n\in\mathbb{N}} F_n$ by $T_{|F_n} = T_n$ for all $n \in \mathbb{N}$. We have for any $x = \sum_{n\in\mathbb{N}} x_n, x_n \in F_n$,

$$\|Tx - x\| \leq \sum_{n\in\mathbb{N}} \|T_n x_n - x_n\| \leq \sum_{n\in\mathbb{N}} \delta_n \|x_n\| \leq \epsilon \|x\|,$$

by bimonotonicity of the basis. $\square$

For $N \in \mathbb{N}$, we let $\mathcal{F}_N(X)$ be the set of elements (F_*, F) of $\mathcal{F}(X)$ such that $\max(ran(F_*, F)) = N$. Fixing $(\delta_n)_{n\in\mathbb{N}}$ a decreasing positive sequence such that $\delta_n \leq 2^{-n}$ for every $n \in \mathbb{N}$, we define $\mathcal{R} \subset \mathcal{F}(X)$ satisfying the following properties:

i) $\mathcal{R} \cap \mathcal{F}_N(X)$ is a finite δ_N-net for $\mathcal{F}_N(X)$,

ii) whenever $(F_{1*}, F_1) < \cdots < (F_{k*}, F_k)$ belong to $\mathcal{R}$, it follows that $(F_{1*} + \ldots + F_{k*}, F_1 + \ldots + F_k)$ belongs to $\mathcal{R}$.

iii) for any $(F_*, F) \in \mathcal{R} \cap \mathcal{F}_N(X)$, $\mathcal{R} \cap \mathcal{F}_{F_*,F}$ is a δ_N-net for $\mathcal{F}_{F_*,F}$, where $\mathcal{F}_{F_*,F}$ denotes $\{(G_*, G) \in \mathcal{F}_N(X) : (G \subset F) \wedge (G_* \subset F_*)\}$.

iv) for any $(F_*, F) \in \mathcal{R} \cap \mathcal{F}_N(X)$, $\mathcal{R} \cap \mathcal{F}_F^{F_*}$ is a δ_N-net for $\mathcal{F}_F^{F_*}$, where $\mathcal{F}_F^{F_*} := \{(G_*, F) \in \mathcal{F}_N(X) : G_* \subset F_*\}$,

v) for any $(F_*, F) \in \mathcal{R} \cap \mathcal{F}_N(X)$, $\mathcal{R} \cap \mathcal{F}_{F_*}^F$ is a δ_N-net for $\mathcal{F}_{F_*}^F$, where $\mathcal{F}_{F_*}^F := \{(F_*, G) \in \mathcal{F}_N(X) : G \subset F\}$,

vi) if $(F_*, F) \in \mathcal{R}$ then $(F^\perp \cap [e_i^*, i \in E], F_*^\perp \cap [e_i, i \in E]) \in \mathcal{R}$, where $E = ran(F_*, F)$.

An *$\mathcal{R}$-block* will denote a block in $\mathcal{R}$. In the following, blocks will always be $\mathcal{R}$-blocks, unless specified otherwise.

We denote by $QS^{<\omega}(X)$ (resp. $QS_0^{<\omega}(X)$) the set of finite sequences of successive $\mathcal{R}$-blocks $(F_{n*}, F_n)_n$ (resp. for which $\min(ran(F_{1*}, F_1)) = 1$).

The set $QS^\omega(X)$ (resp. $QS_0^\omega(X)$) is the space of infinite sequences of successive $\mathcal{R}$-blocks $\mathcal{Y} = (Y_{n*}, Y_n)_n$ (resp. for which $\min(ran(Y_{1*}, Y_1)) = 1$). If $(Y_{n*}, Y_n)_n$ is an element of $QS_0^\omega(X)$, the *partition* of $(Y_{n*}, Y_n)_n$ is the sequence $(ran(Y_{n*}, Y_n))_{n\in\mathbb{N}}$, which forms a partition of $\mathbb{N}$. The space $\sum_{n\in\mathbb{N}} Y_n$ will be denoted Y, and Y_* will denote $\sum_{n\in\mathbb{N}} Y_{n*}$. As was already observed, the relation $Y_*^\perp \subset Y$ ensures that $Y/Y_*^\perp$ is a block-FDD quotient of subspace of X. We let $QS(X) \subset QS_0^\omega(X)$ be the set of sequences which are infinite dimensional, that is such that the QS-space $Y/Y_*^\perp$ is infinite dimensional.

If E is an interval of integers, and $(Y_{n*}, Y_n)_{n\in I}$ is a finite or infinite sequence of successive $\mathcal{R}$-blocks, we shall say that $(Y_{n*}, Y_n)_{n\in I}$ *is well-placed with respect to* E if there exists $m \in I$ such that $\min(ran(Y_{m*}, Y_m)) = \max E + 1$. The set of sequences of $QS(X)$ which are well-placed with respect to E is denoted $QS_E(X)$.

We now define a relation of "tail blocking" on $QS(X)$.

Definition 10 *Let $\mathcal{Z}, \mathcal{Y} \in QS(X)$. If E an interval of $\mathbb{N}$, and $(Z_{i*}, Z_i)_{i\geq p}$ is a blocking of $(Y_{n*}, Y_n)_{n\geq m}$, with $\min(ran(Z_{p*}, Z_p)) = \min(ran(Y_{m*}, Y_m)) = \max E + 1$, then we shall write that $\mathcal{Z} \leq^E \mathcal{Y}$.*

Note that if $\mathcal{Z} \leq^E \mathcal{Y}$ then it follows necessarily that $\mathcal{Z}$ and $\mathcal{Y}$ are well-placed with respect to E. It is also clear that $\leq^E$ a preorder relation, and that $\mathcal{W} \leq^E \mathcal{Y}$ whenever $\mathcal{W}$ and $\mathcal{Y}$ are well-placed with respect to E and $\mathcal{W} \leq \mathcal{Y}$.

We shall need the following easy lemma.

Lemma 11 *Let E be an interval of $\mathbb{N}$, $\mathcal{Y}, \mathcal{Z} \in QS_E(X)$. Assume $\mathcal{Z} \leq^E \mathcal{Y}$. Then there exists $\mathcal{W} \in QS_E(X)$ such that $\mathcal{W} \leq \mathcal{Y}$ and $\mathcal{W} \leq^E \mathcal{Z}$.*

Proof : Let $\min(ran((Y_{m*}, Y_m))) = \max E + 1 = \min(ran((Z_{p*}, Z_p)))$ for some m, p. We define $(W_{n*}, W_n) = (Y_{n*}, Y_n)$ if $n < m$ and $(W_{n*}, W_n) = (Z_{(n-p+m)*}, Z_{n-p+m})$ if $n \geq m$. $\square$

Definition 12 *Let $P \subset QS_E(X)$. We say that P is $\leq^E$-hereditary if whenever $\mathcal{Y} \in P$ and $\mathcal{Z} \leq^E \mathcal{Y}$, then $\mathcal{Z} \in P$. We say that P is $\leq^E$-large if it is $\leq^E$-hereditary and whenever $\mathcal{Y} \in QS_E(X)$, there exists $\mathcal{Z} \leq \mathcal{Y}$ such that $\mathcal{Z} \in P$.*

2.2 A game for QS-pairs

Our proof will be based on an "oriented QS-pairs" Gowers game $G_{\mathcal{A}}^{\mathcal{Y}}$ associated to some subset $\mathcal{A}$ of $QS(X) \times \{-1, 1\}^\omega$ and to some $\mathcal{Y} \in QS(X)$, and defined as follows. Player 1 plays some $\mathcal{W}_1 \leq \mathcal{W}$. Player 2 plays some sign $\epsilon_1 \in \{-1, 1\}$, and some block (U_{1*}, U_1) which is a block of $\mathcal{W}_1$ with $\min(ran(U_{1*}, U_1)) = 1$.

At step n, Player 1 plays some $\mathcal{W}_n \leq \mathcal{Y}$ which is well-placed with respect to $ran(U_{n-1*}, U_{n-1})$. Player 2 plays some sign $\epsilon_n \in \{-1, 1\}$, and some block (U_{n*}, U_n) of $\mathcal{W}_n$ which is successive with respect to (U_{n-1*}, U_{n-1}).

Player 2 wins the game if he produced an infinite sequence $(U_{n*}, U_n, \epsilon_n)_n$ which is in $\mathcal{A}$.

In our application we shall use this game for the set $\mathcal{A}_\delta$ associated to some $\delta > 0$ and defined as the set of $(U_{n*}, U_n, \epsilon_n)_n$ such that there exists $n \in \mathbb{N}$, there exists $u_k \in U_k$, $u_k^* \in U_{k*}$, $1 \leq k \leq n$, such that

$$\left\|\sum_{k=1}^{n} u_k\right\| \left\|\sum_{k=1}^{n} u_k^*\right\| < \delta |\sum_{k=1}^{n} \epsilon_{k-1} u_k^*(u_k)|,$$

where $\epsilon_0 = 1$ is fixed.

A *state* s will be an element of $QS^{<\omega}(X) \times \{-1,1\}^{<\omega}$, where the two sequences are of equal length denoted $|s|$. The set of states will be denoted S. When $\mathcal{Y}$ is well-placed with respect to $(U_{i*}, U_i)_{i<k}$ and $(\epsilon_i)_{i<k}$ is a sequence of signs, we define in an obvious way the game $G_{\mathcal{A}}^{\mathcal{Y}}(s)$, where s is the state $(U_{n*}, U_n, \epsilon_n)_{n<k}$: just rename the steps $1, 2, \ldots$ in the new game step $k, k+1, \ldots$ and then apply the same definition as above; this is the game $G_{\mathcal{A}}^{\mathcal{Y}}$ starting from position s.

If $s = (U_{n*}, U_n, \epsilon_n)_{n<k}$, then $ran(s)$ will denote $ran((U_{n*}, U_n)_{n<k})$, and to simplify the notation we also let $QS_s(X)$ stand for $QS_{ran(s)}(X)$, $\leq^s$ stand for $\leq^{ran(s)}$, "successive to s" mean "successive to (U_{k-1*}, U_{k-1})".

In the following, we fix some subset $\mathcal{A}$ of $QS(X) \times \{-1,1\}^\omega$. Our next definition is the first step of the method of "combinatorial forcing" on $QS(X)$.

Definition 13 *Let s be a state, and let $\mathcal{Y} \in QS_s(X)$.*

The state s accepts $\mathcal{Y}$ if Player 2 has a winning strategy for the game $G_{\mathcal{A}}^{\mathcal{Y}}(s)$.

The state s rejects $\mathcal{Y}$ if it accepts no $\mathcal{Z} \leq \mathcal{Y}$.

The state s decides $\mathcal{Y}$ if it accepts or rejects $\mathcal{Y}$.

Lemma 14 *Let s be a state.*

- the set of $\mathcal{Y}$ in $QS_s(X)$ such that s accepts $\mathcal{Y}$ (resp. rejects $\mathcal{Y}$) is $\leq^s$-hereditary.

- the set of $\mathcal{Y}$ in $QS_s(X)$ such that s decides $\mathcal{Y}$ is $\leq^s$-large.

Proof : Assume s accepts $\mathcal{Y}$. Let $\mathcal{Z}$ be such that $\mathcal{Z} \leq_s \mathcal{Y}$. Let at step n, $\mathcal{W} = \mathcal{W}_n \leq \mathcal{Z}$ be a move for Player 1. By Lemma 11, we may find $\mathcal{V} \leq \mathcal{Y}$ with $\mathcal{V} \leq^s \mathcal{W}$, in particular $\mathcal{V}$ is well-placed with respect to $ran(s)$. Therefore $\mathcal{V}_n = \mathcal{V}$

is an admissible move for Player 1. Since s accepts, a move $(U_{n*}, U_n, \epsilon_n)$ for Player 2 is prescribed by the winning strategy for $G^{\mathcal{Y}}_{\mathcal{A}}(s)$. This move is admissible for Player 2 in $G^{\mathcal{Z}}_{\mathcal{A}}(s)$, since (U_{n*}, U_n) is successive to s and therefore is a block of $\mathcal{W}$. We have therefore described a winning strategy for Player 2 in the game $G^{\mathcal{Z}}_{\mathcal{A}}(s)$, which means that s accepts $\mathcal{Z}$.

Assume now that s rejects $\mathcal{Y} \in QS_s(X)$ while it does not reject $\mathcal{Z} \leq^s \mathcal{Y}$. We may assume that s accepts $\mathcal{Z}$. We get a contradiction by using Lemma 11 to find some element $\mathcal{W} \in QS_s(X)$ such that $\mathcal{W} \leq \mathcal{Y}$ and $\mathcal{W} \leq^s \mathcal{Z}$.

It follows from this that the set of $\mathcal{Y}$ in $QS_s(X)$ such that s decides $\mathcal{Y}$ is $\leq^s$-hereditary. Finally if $\mathcal{Y} \in QS_s(X)$, either s rejects $\mathcal{Y}$, or s accepts some $\mathcal{Z} \leq \mathcal{Y}$; this implies $\leq^s$-largeness. $\square$

Lemma 15 *(stabilization principle) For any $\mathcal{W} \in QS(X)$, there exists $\mathcal{Y} \leq \mathcal{W}$ such that whenever $(Z_{n*}, Z_n)_{n \leq k} \in QS_0^{<\omega}(X)$ is a blocking of $\mathcal{Y}$, and $(\epsilon_n)_{n \leq k}$ is a sequence of signs, it follows that the state $s = (Z_{n*}, Z_n, \epsilon_n)_{n \leq k}$ decides $\mathcal{Y}$.*

Such a $\mathcal{Y}$ will be called stabilizing, and states associated to blockings of $\mathcal{Y}$ will be said to be states blocking $\mathcal{Y}$.

Proof : Let $\mathcal{W}$ be fixed in $QS(X)$. Let n_1 be such that $\dim(W_{n_1*}, W_{n_1}) \geq 1$. We let $\mathcal{Y}^1 = \mathcal{W}$ and let $(Y_{1*}, Y_1) = (\sum_{n \leq n_1} Y^1_{n*}, \sum_{n \leq n_1} Y^1_n)$. Assume $(Y_{k*}, Y_k)_{k<n}$ and some $\mathcal{Y}^{n-1}$ in $QS_E(X)$ were constructed with $E = ran((Y_{n-1*}, Y_{n-1})$. By the finiteness property of $\mathcal{R}$, and the $\leq^E$-largeness property of Lemma 14, we may find some $\mathcal{Y}^n \leq \mathcal{Y}^{n-1}$, with $\mathcal{Y}^n \in QS_E(X)$, such that for any finite sequence $(Z_{i*}, Z_i)_{i \leq m}$ which is a blocking of $(Y_{k*}, Y_k)_{k<n}$ with $\max(ran(Z_{m*}, Z_m)) = \max E$, and for any sequence of signs $(\epsilon_i)_{i \leq m}$, the state $s = (Z_{i*}, Z_i, \epsilon_i)_{i \leq m}$ decides $\mathcal{Y}^n$. Let m_n be such that $\max(ran(Y^n_{m_n*}, Y^n_{m_n}) = \max E$ and p_n be such that the associated subsequence $(Y^n_{i*}, Y^n_i)_{m_n < i \leq p_n}$ contains a term of dimension at least 1. Let (Y_{n*}, Y_n) be $(\sum_{m_n < i \leq p_n} Y^n_{i*}, \sum_{m_n < i \leq p_n} Y^n_i)$.

Repeating this by induction we have constructed an element of $QS(X)$ which satisfies the required property. Indeed for any state s blocking $\mathcal{Y}$, let n be such that $\max(ran(s)) = \max(ran(Y_{n-1*}, Y_{n-1}))$. Then s decides $\mathcal{Y}^n$ and $\mathcal{Y} \leq^s \mathcal{Y}^n$, therefore s decides $\mathcal{Y}$. $\square$

We now fix some stabilizing $\mathcal{X}$ in $QS(X)$. Note that by Lemma 14 and Lemma 15, whenever s is a state blocking $\mathcal{X}$ and $\mathcal{Y} \leq_s \mathcal{X}$, we have that s accepts (resp. rejects) $\mathcal{X}$ if and only if it accepts (resp. rejects) $\mathcal{Y}$. In the following, we shall write s accepts (resp. rejects), to mean that s accepts (resp. rejects) $\mathcal{X}$.

Lemma 16 *Let $s \in S$ be a state blocking $\mathcal{X}$. If s rejects, then for any $\mathcal{Y} \leq \mathcal{X}$ in $QS_s(X)$ there exists $\mathcal{Z} \leq \mathcal{Y}$ in $QS_s(X)$ such that for any (F_*, F) block of $\mathcal{Z}$ which is successive to s, and any sign ϵ, the state $s^\frown(F_*, F, \epsilon)$ rejects.*

Proof : Assume the conclusion is false. Let $n = |s|$. There exists $\mathcal{Y} \leq \mathcal{X}$ in $QS_s(X)$, such that for any $\mathcal{Z} \leq \mathcal{Y}$ in $QS_s(X)$, there is a block (F_{n+1*}, F_{n+1}) of $\mathcal{Z}$ successive to s and $\epsilon_{n+1} \in \{-1, 1\}$ such that the state $s' = s^\frown(F^*_{n+1}, F_{n+1}, \epsilon_{n+1})$ accepts, and therefore accepts $\mathcal{Y}$, that is Player 2 has a winning strategy for $G^{\mathcal{Y}}_{\mathcal{A}}(s')$. Note that s' is a state blocking $\mathcal{X}$. What we wrote means that Player 2 has a winning strategy for $G^{\mathcal{Y}}_{\mathcal{A}}(s)$, in other words s accepts $\mathcal{Y}$, that is s accepts. This is a contradiction. $\square$

In the following $\emptyset$ denote the empty state.

Lemma 17 *Assume $\emptyset$ rejects. Then there exists $\mathcal{Y} \leq \mathcal{X}$ such that any state blocking $\mathcal{Y}$ rejects.*

Proof : Let $\mathcal{Y}^0 = \mathcal{X}$. We build by induction a sequence $\mathcal{Y} = (Y_{n*}, Y_n)_{n\in\mathbb{N}}$ and a $\leq$-decreasing sequence $(\mathcal{Y}^n)_{n\in\mathbb{N}}$ with $\mathcal{Y}^n \in QS_{E_n}(X)$, if $E_n = ran(Y_{i*}, Y_i)_{i<n}$, and with (Y_{n*}, Y_n) a block of $\mathcal{Y}^n$ for each $n \in \mathbb{N}$, as follows. Assume $(Y_{i*}, Y_i)_{i<n}$ and $(\mathcal{Y}^i)_{i<n}$ were defined. There are finitely many states s with $\max(ran(s)) = \max(E)$. Therefore applying Lemma 16 a finite number of times, we obtain some $\mathcal{Y}^n \leq \mathcal{Y}^{n-1}$ in $QS_E(X)$ such that for any state s with $\max(ran(s)) = \max(E)$, for any (F_*, F) block of $\mathcal{Z}$ which is successive to E, and for any sign ϵ, the state $s^\frown(F_*, F, \epsilon)$ rejects. We define (Y_{n*}, Y_n) to be such a block (F_*, F) of dimension at least 1.

Whenever $\mathcal{U} = (U_{n*}, U_n)_{n\in\mathbb{N}} \leq \mathcal{Y}$, we may easily check by induction that for any sequence of signs $(\epsilon_i)_{i\leq n}$, the state $(U_{i*}, U_i, \epsilon_i)_{i\leq n}$ rejects. $\square$

Proposition 18 *Let $\mathcal{A}$ be a subset of $QS(X) \times \{-1, 1\}^\omega$ which is open as a subset of $(\mathcal{F}(X) \times \{-1, 1\})^\omega$ with the product of the discrete topology on $\mathcal{F}(X) \times \{-1, 1\}$. If for every $\mathcal{Y} \in QS(X)$, there exists $\mathcal{Z} \leq \mathcal{Y}$ and a sequence of signs e such that $(\mathcal{Z}, e) \in \mathcal{A}$, then there exists $\mathcal{Y} \in QS(X)$ such that Player 2 has a winning strategy in the game $G^{\mathcal{Y}}_{\mathcal{A}}$.*

Proof : If $\emptyset$ accepts then by definition, Player 2 has a winning strategy in the game $G^{\mathcal{Y}}_{\mathcal{A}}$ for some $\mathcal{Y}$. If $\emptyset$ rejects then, by Lemma 17, there exists $\mathcal{Y}$ of which any

blocking state rejects, which implies that any state blocking $\mathcal{Y}$ is extendable as a sequence which is not in $\mathcal{A}$. Since $\mathcal{A}$ is open, this means that no infinite sequence of successive blocks of $\mathcal{Y}$ and of signs belongs to $\mathcal{A}$. $\square$

Recall that for any $\delta > 0$, we define $\mathcal{A}_\delta$ to be the set of $(U_{n*}, U_n, \epsilon_n)_n$ such that there exists $n \in \mathbb{N}$, and $u_k \in U_k$, $u_k^* \in U_{k*}$, $1 \leq k \leq n$, such that

$$\left\|\sum_{k=1}^{n} u_k\right\| \left\|\sum_{k=1}^{n} u_k^*\right\| < \delta |\sum_{k=1}^{n} \epsilon_{k-1} u_k^*(u_k)|,$$

where we put $\epsilon_0 = 1$. This is an open subset of $(\mathcal{F}(X) \times \{-1,1\})^\omega$.

2.3 A dichotomy theorem on QS(X)

If $\mathcal{Y} \in QS(X)$, with $\dim(Y_{n*}, Y_n) = 1$ for all $n \in \mathbb{N}$, then we shall write that $\mathcal{Y} \in QS_1(X)$. If $\mathcal{Y} \in QS_1(X)$, and for each $n \in \mathbb{N}$, $\tilde{e}_n \in Y/Y_*^\perp$ is the class of some $e_n \in Y_n$ which is not in $Y_{n*}^\perp$, then we shall say that $(\tilde{e}_n)$ is a *successive Schauder basis* of $Y/Y_*^\perp$. Note that all successive Schauder bases of $Y/Y_*^\perp$ may be deduced from each other by homotheties on the span of each of their basic vectors.

In the next proposition, fixing $\delta > 0$, we let $\mathcal{X}_\delta$ be a stabilizing subspace corresponding to $\mathcal{A}_\delta$, and we write s δ-accepts (resp. δ-rejects) to mean that s accepts (resp. rejects) $\mathcal{X}_\delta$ with respect to the set $\mathcal{A}_\delta$.

Proposition 19 *If $\emptyset$ δ-rejects, then there exists $\mathcal{Y} \in QS_1(X)$ with $\mathcal{Y} \leq \mathcal{X}_\delta$, such that any successive basis of $Y/Y_*^\perp$ is unconditional with constant δ^{-1}. If $\emptyset$ δ-accepts, then whenever $\mathcal{U}, \mathcal{V} \leq \mathcal{X}_\delta$ have identical partitions, $A(\mathcal{U}, \mathcal{V}) < \delta$.*

Proof : If $\emptyset$ δ-rejects, then consider $\mathcal{Y} = (Y_{i*}, Y_i)_{i \in \mathbb{N}}$ given by Lemma 17, and write $E_i = ran(Y_{i*}, Y_i)$. Without loss of generality we may assume that $\mathcal{Y}$ belongs to $QS_1(X)$. Pick in each Y_i some normalized f_i such that $d(f_i, Y_{i*}^\perp \cap [e_n, n \in E_i]) = 1$. Fix n and some signs $(\epsilon_i)_{i \leq n}$, and recall that ϵ_0=1. By the proof of Lemma 17, and since $\mathcal{A}_\delta$ is open, we have that for any $(y_i^*, y_i) \in Y_{i*} \times Y_i$, $i \leq n$,

$$\left\|\sum_{k=1}^{n} y_k\right\| \left\|\sum_{k=1}^{n} y_k^*\right\| \geq \delta |\sum_{k=1}^{n} \epsilon_{k-1} y_k^*(y_k)|.$$

Equivalently, whenever $\|\sum_{k=1}^n y_k^*\| = 1$,

$$(\sum_{k=1}^n y_k^*)(\sum_{k=1}^n \epsilon_{k-1} y_k) \leq \delta^{-1} \left\| \sum_{k=1}^n y_k \right\|,$$

and therefore

$$\left\| \sum_{k=1}^n \epsilon_{k-1} y_k \right\|_{\sum_{k\leq n} Y_{k*}} \leq \delta^{-1} \left\| \sum_{k=1}^n y_k \right\|.$$

Taking $y_k = \lambda_k f_k + z_k$, where λ_k is a real number and z_k is arbitrary in $Y_{k*}^\perp \cap [e_i, i \in E_k]$, we obtain

$$\left\| \sum_{k=1}^n \epsilon_{k-1} \lambda_k f_k \right\|_{\sum_{k\leq n} Y_{k*}} \leq \delta^{-1} \left\| \sum_{k=1}^n \lambda_k f_k + z \right\|,$$

where $z \in (\sum_{k\leq n}(Y_{k*}^\perp \cap [e_i, i \in E_k]) = (\sum_{k\leq n} Y_{k*})^\perp \cap [e_i, i \in \cup_{k\leq n} E_k]$ is arbitrary. By duality in $[e_i, i \in \cup_{k\leq n} E_k]$, we conclude that

$$\left\| \sum_{k=1}^n \epsilon_{k-1} \lambda_k f_k \right\|_{\sum_{k\leq n} Y_{k*}} \leq \delta^{-1} \left\| \sum_{k=1}^n \lambda_k f_k \right\|_{\sum_{k=1}^n Y_{k*}}.$$

Since $(\epsilon_i)_{1\leq i\leq n-1}$ was arbitrary, we deduce that $(\tilde{e}_k)_{k\leq n}$ is δ^{-1}-unconditional in $\sum_{k\leq n} Y_k/((\sum_{k\leq n} Y_{k*})^\perp \cap [e_i, i \in \cup_{k\leq n} E_k])$ for each n, and therefore in $Y/Y_*^\perp$ by bimonotonicity.

Assume $\emptyset$ δ-accepts. Pick $\mathcal{U}, \mathcal{V} \leq \mathcal{X}_\delta$ which have identical partitions. This will ensure that playing $\mathcal{U}$ or $\mathcal{V}$ is always an admissible move for Player 1. We therefore may define a strategy for Player 1 as follows. The first move is $\mathcal{U}$. Assuming Player 2 picked some $(Y_{k-1}^*, Y_{k-1}, \epsilon_{k-1})$ at step $k-1$, Player 1's k-th move will be $\mathcal{U}$ if $\epsilon_{k-1} = 1$ and $\mathcal{V}$ if $\epsilon_{k-1} = -1$. Opposing a winning strategy for Player 2, we therefore obtain some $n \in \mathbb{N}$, and some sequences $(u_i^*, u_i)_{i\leq n}$ of pairs of vectors and functionals, and $(\epsilon_i)_{i\leq n}$ of signs such that $u_i \in U, u_i^* \in U_*$ if $\epsilon_{i-1} = 1$ and $u_i \in V, u_i^* \in V_*$ if $\epsilon_{i-1} = -1$, and with

$$\left\| \sum_{k=1}^n u_k \right\| \left\| \sum_{k=1}^n u_k^* \right\| < \delta |\sum_{k=1}^n \epsilon_{k-1} u_k^*(u_k)|.$$

We let $u = \sum_{\epsilon_{i-1}=1} u_i \in U$, $u^* = \sum_{\epsilon_{i-1}=1} u_i^* \in U_*$, $v = -\sum_{\epsilon_{i-1}=-1} u_i \in V$, $v^* = -\sum_{\epsilon_{i-1}=-1} u_i^* \in V_*$, and observe that $u^*(v) = v^*(u) = 0$ and

$$\|u - v\| \, \|u^* - v^*\| < \delta |u^*(u) - v^*(v)|.$$

$\square$

Theorem 20 *Let X be a Banach space with a Schauder basis. Then there exists a quotient of subspace $Y/Y_*^{\perp}$ of X, associated to some $\mathcal{Y}$ in $QS_1(X)$, which satisfies one of the two following properties, which are both possible and mutually exclusive:*
i) $Y/Y_*^{\perp}$ has an unconditional basis,
ii) $Y/Y_*^{\perp}$ has the restricted QHI property.

Proof : Fix as before a positive sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n \leq 2^{-n}$ for all n, and build by Lemma 15 a $\leq$-decreasing sequence $\mathcal{X}_n$ such that $\mathcal{X}_n$ is δ_n-stabilizing for each n. If, with the notation defined as the beginning of this subsection, $\emptyset$ δ_n-rejects $\mathcal{X}_n$ for some n, then we are done by Proposition 19.

Assume therefore that $\emptyset$ δ_n-accepts $\mathcal{X}_n$ for all $n \in \mathbb{N}$. Let $\mathcal{Y} \in QS(X)$ be diagonal for the $\mathcal{X}_n$'s, i.e. such that for any state s blocking $\mathcal{Y}$, with $\max(ran(s)) = \max(ran(Y_{n*}, Y_n))$, we have that $\mathcal{Y} \leq^s \mathcal{X}_n$. This is easily constructed by induction. We shall prove that $A(\mathcal{U}, \mathcal{V}) = 0$ for any $\mathcal{U}, \mathcal{V} \leq \mathcal{Y}$ which are sequences of successive blocks (not necessarily in $\mathcal{R}$) with equal ranges forming a partition of $\mathbb{N}$. By Lemma 8, this will be enough to prove our result.

Fix $\epsilon > 0$, and arbitrary $\mathcal{U}, \mathcal{V} \in QS(X)$ (therefore formed of $\mathcal{R}$-blocks), with $\mathcal{U}, \mathcal{V} \leq \mathcal{Y}$ and with identical partition denoted $(E_n)_{n\in\mathbb{N}}$. Let m be large enough so that if $p = \max(E_m)$ then $\delta_p < \epsilon$. Denote by $\mathcal{X} = (X_{i*}, X_i)_{i\in\mathbb{N}}$ the corresponding $\mathcal{X}_p$, by F_i the range of (X_{i*}, X_i) and let q be such that $p = \max(F_q)$.

We let for $i \leq q$, $U'_{i*} = X_{i*}$, and $U'_i = X_{i*}^{\perp} \cap [e_n, n \in F_i]$. For $i > q$ we let $(U'_{i*}, U'_i) = (U_{m-q+i*}, U_{m-q+i})$. We have therefore constructed an element $\mathcal{U}'$ of $QS(X)$ which satisfies $\mathcal{U}' \leq \mathcal{X}$ and $\mathcal{U}' \leq^E \mathcal{U}$ for $E = [1, p]$. We construct in the same way some $\mathcal{V}' \leq \mathcal{X}$, $\mathcal{V}' \leq^E \mathcal{V}$.

By Proposition 19, we may find $x, x^* \in \mathcal{U}'$ and $y, y^* \in \mathcal{V}'$, with disjoint supports, with $\|x - y\| \, \|x^* - y^*\| < \delta_p |(x^* - y^*)(x + y)|$. Let P be the projection onto $[e_n, n > p]$, and P_* be the projection onto $[e_n^*, n > p]$. Note that $P(U') \subset U$ and $P_*(U'_*) \subset U_*$, and the similar inclusions hold for V and V_*. Let $u = Px, u^* = P_*x^*, v = Py, v^* = P_*y^*$.

By bimonotonicity of the basis, we observe that $\|u-v\| \leq \|x-y\|$ and $\|u^*-v^*\| \leq \|x^*-y^*\|$. On the other hand, writing $x = u+a, x^* = u^*+a^*, y = v+b, y^* = v^*+b^*$, we note that $a \in (\sum_{i\leq q} X_{i*}^{\perp}$ while $a^* \in \sum_{i\leq q} X_{i*}$, therefore $a^*(a) = 0$. Likewise, $b^*(b) = a^*(b) = b^*(a) = 0$, and by disjointness of the ranges, $u^*(a) = u^*(b) = v^*(a) = v^*(b) = a^*(u) = a^*(v) = b^*(u) = b^*(v) = 0$. Therefore

$$(x^*-y^*)(x+y) = (u^*-v^*)(u+v),$$

and we deduce that

$$\|u-v\|\,\|u^*-v^*\| < \delta_p|(u^*-v^*)(u+v)|.$$

We have therefore proved that $a(\mathcal{U},\mathcal{V}) < \epsilon$.

It remains to show that we may obtain the same results for general $\mathcal{U},\mathcal{V} \leq \mathcal{Y}$, i.e. successive sequences of blocks which are not necessarily in $\mathcal{R}$. Fix $0 < \epsilon < 1/3$ and let $\mathcal{U}$, $\mathcal{V}$ have the same partition $(E_n)_{n\in\mathbb{N}}$. Let $\mathcal{U}'$, $\mathcal{V}'$ be sequences with blocks in $\mathcal{R}$, such that $\delta((U_{n*},U_n),(U'_{n*},U'_n)) < \delta_n$ for all $n \in \mathbb{N}$, and the similar relations for (V_{n*},V_n) and (V'_{n*},V'_n). Let $N \in \mathbb{N}$ be such that $2^{-N} \leq \epsilon/2$. By the above, we may find a partition $\{I,J\}$ of $[N,+\infty)$, vectors $u \in \sum_{n\in I} U'_n$, $v \in \sum_{n\in J} V'_n$, and functionals $u^* \in \sum_{n\in I} U'_{n*}$, $v^* \in \sum_{n\in J} V'_{n*}$, such that $\|u-v\|\,\|u^*-v^*\| < \epsilon|(u^*-v^*)(u+v)|$.

Let for $n \geq N$, $(W_{n*},W_n) = (U_{n*},U_n)$ if $n \in I$, and $(W_{n*},W_n) = (V_{n*},V_n)$ if $n \in J$, and let $W_* = \sum_{n\geq N} W_{n*}$; let (W'_{n*},W'_n) and W'_* be defined in a similar way. Let also $X_N = [e_i, i \in \cup_{n\geq N} E_n]$. We have therefore

$$\|u-v\| < \epsilon\,\|u+v\|_{W'_*}.$$

Since $\sum_{n\geq N}\delta_n \leq \epsilon$, we find by Lemma 9 a map T from $\sum_{n\geq N} W'_n$ onto $\sum_{n\geq N} W_n$ such that $T(W') = W$, $T(X_N \cap {W'_*}^{\perp}) = X_N \cap W_*^{\perp}$, and with $\|T\|\|T^{-1}\| \leq (1+\epsilon)(1-\epsilon)^{-1} \leq 2$. If let $x = Tu \in \sum_{n\in I} U_n$ and $y = Tv \in \sum_{n\in J} V_n$, we have

$$\|x-y\| < 2\epsilon\,\|x+y\|_{W_*}.$$

This means that we may pick some normalized functional $w^* \in W_*$, therefore $w^* = x^*-y^*$ with $x^* \in \sum_{n\in I} U_{n*}$, $y^* \in \sum_{n\in J} V_{n*}$, such that

$$\|x-y\| < 2\epsilon|(x^*-y^*)(x+y)| = 2\epsilon|x^*(x)-y^*(y)|,$$

and we deduce that $A(\mathcal{U},\mathcal{V}) \leq 2\epsilon$. $\square$

3 Remarks and open questions

Remark 21 Let Y be an FDD-block quotient of subspace of X. To check whether Y has the restricted QHI property, we have checked the formally stronger result that the angle is zero between any two QS-pairs associated to FDD-block quotient of subspaces of Y with sequences of blocks having the same partition. We note that by our dichotomy theorem, these two notions are equivalent up to passing to a quotient of subspace. Indeed if X is QHI restricted to block-subspaces, then no quotient of X by an FDD-block subspace can contain an unconditional basic sequence, and therefore X must contain a quotient of subspace with the stronger "angle zero" property.

Question 22 Is it possible to improve Theorem 20 to suppress the restriction to FDD-block subspaces? The restricting condition is not only technical. By a result of S. Argyros, A. Arvanitakis and A. Tolias [1], the distinction between general quotient spaces and quotient by FDD-block subspaces can be essential: there exists a separable dual space X with a Schauder basis, such that quotients with w^*-closed kernels are HI, yet every quotient has a further quotient isomorphic to l_2. Since FDD-block subspaces of X are w^*-closed, this space has the restricted QHI property, but it is not QHI by the l_2-saturation property.

Contrary to the case of subspaces, it does not seem clear that, in a space with a Schauder basis, QS-spaces may be approximated by FDD-block quotient of subspaces, that is, that for any QS-space, there is a further QS-space, which is an arbitrary small perturbation of an FDD-block quotient of subspace.

Remark 23 As was noticed in the introduction, HI spaces can fall in either side of the dichotomy in Theorem 6. The example of X_{GM} is QHI, while the examples of [2] have an unconditional quotient. The dual X_{uh}^* of the reflexive space X_{uh} of Argyros and Tolias [5] has the following quite interesting mixed property. Any of its quotients has a further quotient with an unconditional basis, [5] Proposition 3.6. On the other hand it is HI, [5] Proposition 5.11, and it is saturated with QHI subspaces. This last fact was indicated to us by S. Argyros and the proof is as follows. Consider any block subspace of X_{uh}^*. Keeping a half of the vectors of the block basis, and denoting the space generated by them Y, we get that the annihilator of any subspace Z of Y must contain an infinite subsequence of the basis. Therefore [5] Proposition 6.3. applies to obtain that $X_{uh}/Z^{\perp}$ is HI. This means that every infinite dimensional quotient of $X_{uh}/Y^{\perp}$ is

HI, therefore $X_{uh}/Y^{\perp}$ is QHI. By reflexivity it follows that $Y \simeq (X_{uh}/Y^{\perp})^*$ is QHI.

We say that a Banach space X is *unconditionally QS-saturated (resp. QS-saturated with HI subspaces)* if any infinite dimensional QS-space of X has a further QS-space with an unconditional basis (resp. which is HI).

By Odell's result [13], if a space X has a shrinking unconditional FDD, then every quotient of X must be unconditionally saturated, and therefore X must be unconditionally QS-saturated. It remains unknown whether there exists a HI space which is unconditionally QS-saturated. Therefore we ask:

Does every HI space contain a quotient of subspace which is QHI? or which has the restricted QHI property?

Remark 24 Our dichotomy theorem, the result of Odell [13], and the remark after Proposition 5 imply the following: if X is reflexive and QS-saturated with HI spaces, then some quotient of subspace of X is saturated with subspaces with HI dual.

In this direction, we recall the question of S. Argyros:

Does there exist a reflexive HI space X, such that no subspace of X has a HI dual?

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